

# Dimensions, maximal growth sites, and optimization in the dielectric breakdown model

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We study the growth of fractal clusters in the dielectric breakdown model (DBM) by means of iterated conformal mappings. In particular we investigate the fractal dimension and the maximal growth site (measured by the Hoelder exponent  $\alpha_{min}$ ) as a function of the growth exponent  $\eta$  of the DBM model. We do not find evidence for a phase transition from fractal to nonfractal growth for a finite  $\eta$  value. Simultaneously, we observe that the limit of nonfractal growth ( $D \rightarrow 1$ ) is consistent with  $\alpha_{min} \rightarrow 1/2$ . Finally, using an optimization principle, we give a recipe on how to estimate the effective value of  $\eta$  from temporal growth data of fractal aggregates.

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## I. INTRODUCTION

Laplacian growth and the formation of complex patterns has been the subject of numerous theoretical and experimental works. The classical examples are the ramified pattern appearing in a Hele-Shaw cell when a less viscous fluid is injected into a more viscous fluid [1] and the fractal structures emerging from the particle aggregation in diffusion-limited aggregation (DLA) [2]. In the latter example monodisperse particles are released one-by-one from a remote source and diffuse until they hit and irreversibly adhere to a seed cluster at the center of coordinates. The cluster slowly expands as particles are added. Statistically, the motion of a single particle is described by the harmonic potential  $U$  satisfying the Laplace equation  $\Delta U = 0$  and the probability of sticking to the cluster at a specific site,  $z$ , is given by the harmonic measure  $|\nabla U(z)|$ . The formulation of DLA is contained within a more general model, the dielectric breakdown model (DBM) [3,4], where the growth probability  $\rho_\eta$  at the cluster interface is proportional to the harmonic measure raised to a power  $\eta$ ,  $\rho_\eta \propto |\nabla U|^\eta$ . Despite intensive research in Laplacian growth, fundamental questions regarding the scaling properties still have no answer. The growth laws of DLA and DBM are extremely simple and in apparent disparity to the complex patterns they produce. The complex patterns arise from a strong correlation between the position of already aggregated particles and the influx of new particles. As the outermost tips advance, the probability for particles to reach the parts left behind diminishes and the harmonic measure broadens and becomes even multifractal [5]. For increasing values of  $\eta$  the growth probability will concentrate around the tips and the fractal dimension gets closer to unity and ultimately, in the limit of infinite  $\eta$ , the particle cluster loses fractality. Recently, it has been speculated that in two dimensions this transition from fractal to nonfractal growth may happen at a finite critical value of  $\eta$  and numerically, this value has been found to be  $\eta \approx 4$  [4,6]. In the vicinity of

such a critical point it may be safe to disregard the noise giving rise to local density fluctuations along the branches [7]. For that reason, the dominating stochastic component in the cluster growth is the rate at which growing tips split in two or more branches. While growing, neighboring branches compete, and if one branch quickly dies after a tip splitting the growth will stay nonfractal. It has been shown [8] that in the idealized case of straight growing branches, tip splitting is suppressed for  $\eta > 4$  supporting that  $\eta_c = 4$ . Based on the idealized branch growth model a renormalization group approach has been used in an expansion around  $\eta_c$ . Although an expansion provides important information for small values of  $4 - \eta$  it may provide little information on DLA ( $\eta = 1$ ). In this paper we test the hypothesis of a critical point at  $\eta = 4$  performing extensive numerical simulations. We provide detailed figures on the dependence of the fractal dimension,  $\alpha_{min}$  and the exponent  $\eta$ . Moreover, we propose a method for extracting effective  $\eta$  exponents given either experimental or numerical data series. For that purpose we make use of iterated conformal maps [9] which have proven a convenient tool for generating conformal mappings of domains of arbitrary shape [10]; see Sec. II. In Sec. III a method is proposed for extracting effective  $\eta$  exponents by optimization. In Sec. IV we present results *pro et con* a phase transition in DBM, the maximal growth sites, and the fractal dimensions.

## II. ITERATED CONFORMAL MAPPINGS

The conformal invariance of the Laplace equation reduces the problem of finding the harmonic measure,  $\rho_1(z)$ , around any simply connected domain in the complex plane to that of finding a conformal transformation  $\omega = \Phi^{-1}(z)$  of the domain to the unit disc

$$\rho_1(z) = \frac{1}{|\Phi'(w)|}. \quad (1)$$

The method of iterated conformal mappings provides a general framework to construct such transforms as well as a simple procedure to grow DLA clusters. Assume that a DLA

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cluster of  $n$  particles is mapped to the unit disc by  $\Phi_n^{-1}$ . An extra particle is added to the cluster by first adding a small bump of size  $\sqrt{\lambda}$  to the unit disc using a mapping  $\varphi_{n+1}$  and subsequently applying the inverse mapping  $\Phi_n$ . Finally, the composed mapping  $\Phi_n \circ \varphi_{n+1}$  transforms the unit disc into a cluster of  $n+1$  particles. The basic mapping  $\varphi_{n+1}$  is defined by two parameters the position and size of the bump, the position,  $e^{i\theta}$ , is random in DLA since the measure is uniform around the circle. The size  $\sqrt{\lambda}$  of the  $n$ th bump is controlled by the condition that

$$\sqrt{\lambda_0} = \sqrt{\lambda_n} |\Phi'(e^{i\theta})|.$$

Consequently, the particles (transformed bumps) will all to linear order have the same size  $\sqrt{\lambda_0}$ . The full recursive dynamics is written as iterations of the basic map

$$\Phi^{(n)}(w) = \varphi_{\theta_1, \lambda_1} \circ \dots \circ \varphi_{\theta_n, \lambda_n}(w). \quad (2)$$

Note that this structure is unusual in the sense that the order of iterates is inverted compared to standard dynamical systems.

For DBM the growth measure along the cluster interface, parameterized by  $s$ , is given by

$$\rho_\eta(s) = \frac{\rho_1^\eta(s)}{\int \rho_1^\eta(t) dt}, \quad (3)$$

which for  $\eta \neq 1$  is not conformally invariant. On the unit circle, parameterized by  $\theta$ , the growth measure transforms into

$$\rho_\eta(\theta) d\theta \sim \rho_\eta(s(\theta)) \left| \frac{ds}{d\theta} \right| d\theta \sim |\Phi'(e^{i\theta})|^{1-\eta} d\theta. \quad (4)$$

In the simulations we choose  $\theta$  according to the distribution  $\rho_\eta$  using standard Monte Carlo samplings of the measure. The number of samples needed for an accurate estimate of the distribution increases with  $\eta$  and is chosen according to

$$\frac{k}{\sqrt{\lambda_0} \max_s \rho_1(s)}. \quad (5)$$

By choosing  $k > 1$ , the site of maximal measure will on the average be visited more than once during the sampling. It turns out that there is no visible change in the scaling of the clusters when choosing  $k > 1$ . See Fig. 1 for a test of convergence as a function of  $k$ ; in the results presented below, we use  $2 \leq k \leq 8$ .

### III. EXTRACTING EFFECTIVE $\eta$ EXPONENTS BY OPTIMIZATION

Consider an interface growing at a rate determined by some unknown function of the harmonic measure. The method of iterated conformal mappings is readily turned into a framework for estimating this function. More specifically, it is here demonstrated on numerical simulation data of the DBM that the value of  $\eta$  can be extracted from a careful tracking of the cluster growth. The general idea is to utilize

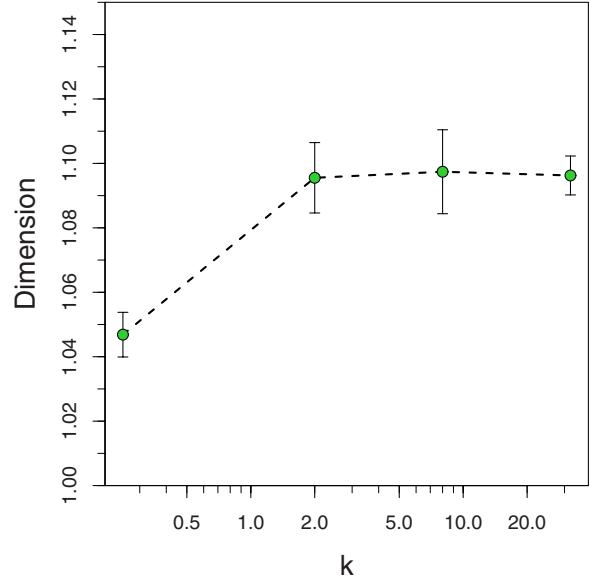


FIG. 1. (Color online) Test of convergence for the deployed Monte Carlo method. Estimates of the fractal dimension for  $\eta=4$  and for Monte Carlo samples given by  $k=1/4, 2, 8, 32$  in Eq. (4). For each data point, we used four clusters of size 40000 particles.

the iteration scheme in tracking the motion of the interface by gradually expanding the mapping, see [10] for further details. The harmonic measure is recorded as the interface evolves, and from a maximum likelihood principle the  $\eta$  value of the growth is extracted. The probability for growth to occur at a site  $z_n$  at the interface is in a given growth step  $n$  approximated by the sum

$$\rho_\eta(n, z_n) = \frac{\rho_1^\eta(n, z_n)}{\sum_z \rho_1^\eta(n, z)}. \quad (6)$$

From this expression, more ways exist to estimate the  $\eta$  value used in the simulation. Assuming that the  $n$ th growth event occurred at the site  $z_n$ , a direct estimate of  $\eta$  follows from maximizing  $\rho_\eta(z_n)$  with respect to  $\eta$ . Naturally, this will lead to dramatic fluctuations in the estimates and therefore maximizing products of  $\rho_\eta$  over several growth steps provides a better estimate,

$$\prod_k \rho_\eta(k, z_k). \quad (7)$$

In Fig. 2, we show how this product varies as function of  $\eta$  and with the number of factors used. With an increasing number of factors the maximum becomes more pronounced and the  $\eta$  value used in the simulations is easily recovered. These products confirm that the number of Monte Carlo samples used in Eq. (5) are appropriate and, more importantly, that the method is directly applicable to experimental data for estimating an effective  $\eta$  value or more generally the boundary condition function determining the growth rate.

### IV. DIMENSION AND $\alpha_{min}$

The dimension of a cluster grown by this conformal mapping technique is determined by the first term in the Laurent

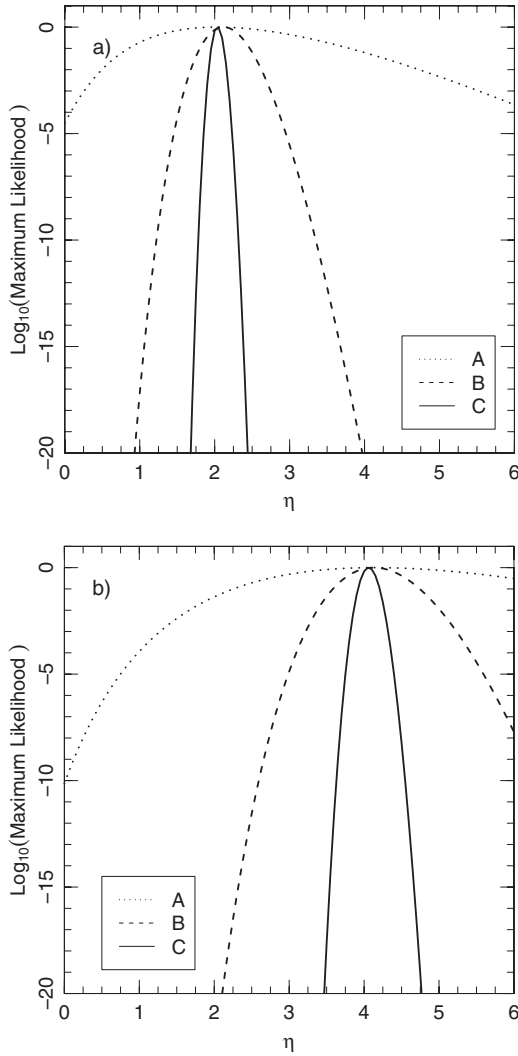


FIG. 2. Product of the growth measure, maximum likelihood,  $\prod_{k=1}^N \rho_\eta(2k, z_{2k})$  as a function of  $\eta$ , for  $N=15, 150, 1500$  (A, B, and C, respectively). The clusters applied were grown with (a)  $\eta=2$  and (b)  $\eta=4$ , consistent with the extreme values of the maximum likelihood. Note that the products were normalized by their maximum value and that every second growth step was used only. The latter was done to avoid the products becoming too small.

expansion of  $\Phi^{(n)}$ ,  $F_1^{(n)}$ , which will scale like  $F_1^{(n)} \sim n^{1/D} \sqrt{\lambda_0}$  [9]. The dimension is thus estimated by a direct fit of this scaling law as demonstrated in Fig. 3 for a cluster 80000 particles and  $\eta=4.0$ .

Using the conformal mapping technique we have grown clusters up to sizes 80000 particles with varying values of  $\eta$  in the interval  $\eta \in [1, 5]$ . Figure 4 shows the results for the value of the dimension versus  $\eta$ . As is clear from the figure, the value of the dimension decreases smoothly with  $\eta$ , from the DLA value  $D=1.71$  for  $\eta=1$  down towards  $D \sim 1$  for  $\eta \rightarrow \infty$ . Hastings [8] presented arguments in favor of an upper critical dimension  $\eta_c=4$  for which the clusters become one-dimensional. We however do not observe indications of this transition. As seen in Fig. 4 it is quite clear that the data smoothly bends away before reaching the point  $(\eta, D) = (4, 1)$  and only approaching the one-dimensional growth in

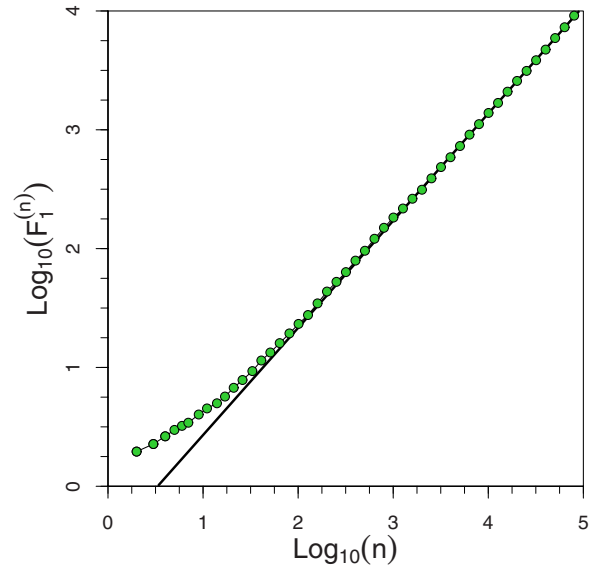


FIG. 3. (Color online) First Laurent coefficient versus cluster size for  $\eta=4$  and 80000 particles (with  $k=2$ , see text). The added line is a fit of the fractal dimension  $D=1.10$ .

the limit of large  $\eta$  values. We thus conclude that a critical point at a finite  $\eta$  does not exist.

Halsey [7] has computed a first-order correction to  $D$  for  $\eta < 4$ , obtaining  $D = 1 + \frac{1}{2}(4 - \eta) + O(4 - \eta)^2$ . This relation predicts a linear variation of slope  $\frac{1}{2}$  around  $\eta_c=4$ . As seen in Fig. 4 we do not observe this behavior.

It is well known that the growth measure of a DBM model exhibits multifractal properties with a spectrum of growth exponents measured by local Hoelder exponents  $\alpha$

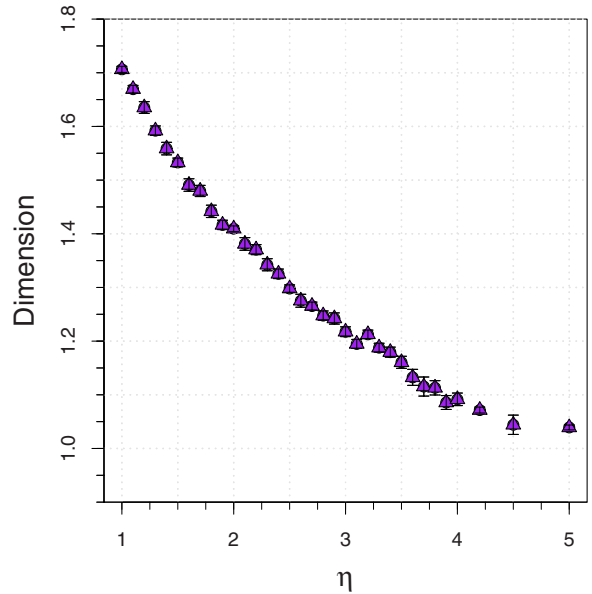


FIG. 4. (Color online) Computed values of the fractal dimension versus  $\eta$ . The dimension was extracted from a fit of the first Laurent coefficient  $F_1$  using clusters of sizes 20000–40000 particles for  $\eta \leq 3.5$  and 40000–80000 particles for  $\eta > 3.5$ . Each data point is averaged over 20 clusters and the error bars are estimated by the standard deviation.

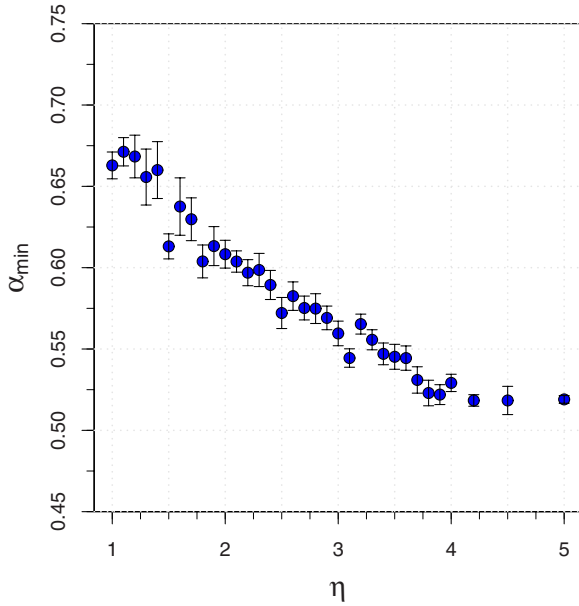


FIG. 5. (Color online) Computed values of  $\alpha_{min}$  versus  $\eta$  using the same clusters as in Fig. 4.

[5]. The points of highest growth measures are characterized by the minimum  $\alpha$  value,  $\alpha_{min}$ . We have previously determined this value using the iterated conformal mapping technique [11] and extend it here to the DBM model. In this method, it is very easy to keep track of where the maximum growth probability is located as more particles are added. Let us assume that at the  $(n-1)$ th growth step the site with the largest probability is located at the angle  $\theta_{max}$  on the unit circle, i.e., for all  $\theta$

$$\frac{1}{|\Phi^{(n-1)'}(e^{i\theta_{max}})|} \geq \frac{1}{|\Phi^{(n-1)'}(e^{i\theta})|}. \quad (8)$$

When we add a new bump in the  $n$ th growth step the position of maximal probability may not change (up to reparametrization of the angle  $\theta_{max}$ ), or move to the new bump. We can easily find the reparametrized angle and determine the new position from

$$\rho_1^{max,n} = \max \left\{ \frac{1}{|\Phi^{(n)' }(\phi_{\lambda_n, \theta_n}^{-1}(e^{i\theta_{max}}))|}, \frac{1}{|\Phi^{(n)' }(e^{i\theta_n})|} \right\}. \quad (9)$$

If  $\rho_1^{max,n}$  is located at  $\theta_n$  we put  $\theta_{max}=\theta_n$  in the  $(n+1)$ th growth step. Using conformal mappings, we have also previously estimated the critical branching angle as a function of  $\eta$  in the DBM model [12].

Figure 5 shows the results of  $\alpha_{min}$  vs  $\eta$  and we observe that  $\alpha_{min}$  decreases from the DLA values  $\alpha_{min}=0.68$  down to  $\alpha_{min}=0.5$ . It is obvious that  $\alpha_{min}=0.5$  corresponds to the Hoelder exponent for a line. In consistency with the results in Fig. 4 we observe that the curve bends smoothly and that the one-dimensional growth is only obtained in the limit  $\eta \rightarrow \infty$ . The last figure, Fig. 6, shows  $\alpha_{min}$  plotted vs  $D$ . By

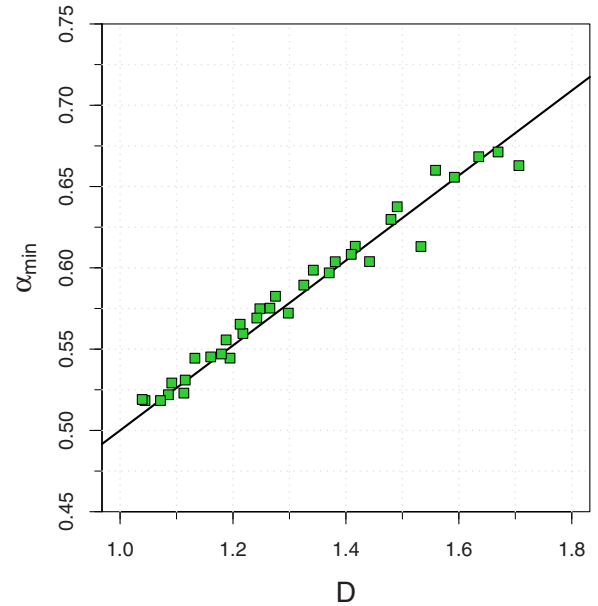


FIG. 6. (Color online)  $\alpha_{min}$  versus the fractal dimension  $D$ . The data are the same as used in Figs. 4 and 5.

extrapolation (as indicated by the line) we see that  $\alpha_{min}$  assumes its minimal value 0.5 at a dimension  $D=1.0$ .

### V. CONCLUSIONS

The conclusions of our paper are twofold. First, we have presented a method to extract the effective value of the growth exponent  $\eta$ , for a time series of growing aggregates, assuming an underlying mechanism based on the dielectric breakdown model (DBM). The estimate is based on a maximum likelihood method and converges rather well for the numerical data presented here. We believe this method should be directly applicable to experimental data when it is possible to extract intermediate steps in the formation of the aggregates. We urge the method to be used in, for example, viscous fingering experiments in random media [13]. Second, we have thoroughly investigated the scaling structure of DBM clusters as a function of the growth exponent  $\eta$ . Based on extensive numerical simulations we do not find support for the conjecture that the growth becomes one-dimensional at the critical value  $\eta_c=4$  [7,8]. On the contrary, our results indicate that a critical point at finite  $\eta$  value does not exist, and that the scaling exponent of the maximal growth site  $\alpha_{min}$  assumes its minimal value 0.5 when the growth becomes nonfractal.

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